Midterm - Optimization (2023-24)
Time: 3 hours.
Attempt all questions. The total marks is 30.
You may quote any result proved in class without proof.

1. Consider the following linear program: minimize $-x_1 - 2x_2 + x_3 + 2x_4 - 6x_5$ subject to $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \leq 0,\ x_4 \leq 0$ and x_5 free, and

$$\begin{array}{cccccc} -x_1 & & -x_3 & +x_4 & -x_5 & \geq 1 \\ & -x_2 & +x_3 & +4x_4 & -2x_5 & \geq 3. \end{array}$$

   (a) Find the dual of the above problem. **[2 marks]**

   (b) Find the optimal cost of the above problem. **[2 marks]**

2. Consider the problem: minimize $2x_1 + 3|x_2 - 10|$ subject to $x_1 + x_2 \leq 5$. Reformulate this as a linear programming problem. **[2 marks]**

3. A polyhedron is represented by a system of equality and inequality constraints.

   (a) Give an example of a polyhedron P and a point $\mathbf{x}$, and two representations of P, such that $\mathbf{x}$ is a basic solution under one representation but is not a basic solution under the other representation. **[2 marks]**

   (b) Can one create an example of P with two representations, and an $\mathbf{x}$ such that $\mathbf{x}$ is a basic *feasible* solution (bfs) under one representation and not a bfs under the other representation? Explain. **[2 marks]**

4. Suppose that $\{\mathbf{x} \in \mathbf{R}^n : \mathbf{a}_i^T\mathbf{x} \geq b_i,\ i = 1, 2\cdots m\}$ and $\{\mathbf{x} \in \mathbf{R}^n : \mathbf{g}_i^T\mathbf{x} \geq h_i,\ i = 1, 2\cdots, k\}$ are two representations of the same nonempty polyhedron. Suppose that the vectors $\mathbf{a}_1, \cdots \mathbf{a}_m$ span $\mathbf{R}^n$. Show that the same must be true for the vectors $\mathbf{g}_1, \cdots, \mathbf{g}_k$. **[5 marks]**

5. Solve the following linear program. **[5 marks]**

$$\begin{array}{rccccc} \text{minimize} & -3x_1 & -2x_2 & +5x_3 & \\ \text{such that} & 4x_1 & -2x_2 & +2x_3 & \leq 4, \\ & 2x_1 & -x_2 & +x_3 & \leq 1, \\ \text{and all} & x_1, & x_2, & x_3 & \geq 0. \end{array}$$

6. Consider the *uncapacitated* network flow problem on the directed graph in the following page. The numbers next to each directed arc $\rightarrow$ is the *cost* associated to the arc, while the numbers next to $\Rightarrow$ is the external *supply/demand* at the node.

   Denote by $\mathbf{c}$ the vector of costs corresponding to the arc. We are interested in minimizing the total cost $\mathbf{c}^T\mathbf{f}$, where the flow vector $\mathbf{f}$ satisfies the flow conservation equations and $\mathbf{f} \geq \mathbf{0}$.

   (a) Find an optimal basic feasible solution (feasible tree solution) to the problem. (You can start with the tree given by the dotted lines) **[6 marks]**

   (b) Find the optimal cost for the problem. **[1 marks]**

   (c) Find the dual vector which gives the same optimal cost. **[3 marks]**